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CHIRAL GAUGE THEORIES ON THE LATTICE*

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We propose a method for constructing lattice gauge theories in which fermions transform as a complex representation of the gauge group.

1. Introduction

The behavior of asymptotically free gauge theories with massless fermions transforming as a complex representation of the gauge group has been the subject of much speculation [1]. Especially intriguing is the suggestion that many such theories are likely to contain massless composite fermions [2].

We will refer to a gauge theory with fermions transforming as a complex representation of the gauge group as a "chiral gauge theory", because the gauged symmetry is a chiral symmetry, rather than a vector-like symmetry as in QCD. That is, the gauge symmetry forbids masses for at least some of the elementary fermions. Examples are the standard SU(3) × SU(2) × U(1) model and the grand unified SU(5) model.

A technical obstacle, the fermion doubling problem, has prevented the construction of chiral gauge theories on the lattice. It is important to surmount this obstacle, both to ensure that continuum chiral gauge theories really exist, and to provide a framework for doing nonperturbative calculations in these theories. In this paper, we propose a new way of dealing with the fermion doubling problem in a chiral gauge theory. Briefly, the basic idea of our approach is that the unwanted "mirror fermions" can acquire large masses consistent with gauge invariance by pairing up with *composite* fermion states with appropriate gauge quantum numbers. These composite fermions may be bound, not by the gauge interaction itself, but by an auxiliary interaction introduced for this explicit purpose. Thus, the mirror fermions can be forced to decouple as the continuum limit is approached.

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Nonperturbative calculations in chiral gauge theories are important because they are necessary to answer the central dynamical question concerning these theories, which may be formulated as follows: Consider an asymptotically free gauge theory with gauge group G and with massless left-handed Weyl fermions transforming as some complex representation of G. If the fermion representation is reducible, then this theory respects a group G_f of global flavor symmetries. We wish to know how the G_f symmetry is realized. This question has two parts: (i) What subgroup H_f of G_f escapes spontaneous symmetry breakdown? (ii) What is the representation content under H_f of the massless fermions in the spectrum of the theory?

The massless fermions may be either composite or elementary, for the G gauge interaction may or may not be exactly confining. (Indeed, finding the realization of the gauge symmetry is a second very important dynamical problem.) If, for example, the gauge group G is spontaneously broken to a subgroup H, massless elementary fermions may appear in the spectrum which are H singlets, but not G singlets. If, however, the G interaction confines, then all physical states, including any massless fermions, must be G-singlet composite states. Whether elementary or composite, the massless fermions are prevented from acquiring masses by the unbroken H_f flavor symmetry.

Finding the realization of the global G_f symmetry is equivalent (if we ignore possible discrete subgroups of G_f and H_f and possible "accidental" massless particles) to identifying the massless spectrum of the theory, since a (necessarily composite) Goldstone boson is associated with each spontaneously broken generator of G_f. It is a problem with a qualitative rather than a quantitative solution. Nonetheless it is a difficult strong coupling problem, to which the solution is not known.

An important step toward solving this problem was taken by 't Hooft [3], who argued that the massless fermions in the spectrum must obey a remarkable algebraic condition – they must produce the same triangle anomalies for the unbroken flavor group H_f as the elementary fermions. This condition places constraints on the H_f representation content of the massless fermions that, in a confining theory, are highly nontrivial, because G-singlet composite fermions must typically be in different representations of H_f than the elementary fermions.

The 't Hooft anomaly condition alone, however, does not uniquely determine the realization of the global G_f symmetry, even if exact confinement is assumed. Further constraints are needed to determine which of the many realizations of G_f allowed by the anomaly condition is actually picked out by the dynamics of a particular chiral gauge theory. For vector-like theories such as QCD, we can appeal to rigorous inequalities [4] or to the behavior of the theory in the limit of a large number of colors [5] to establish that global chiral symmetries are spontaneously broken. But analogous inequalities have not been derived for chiral gauge theories, and it has not yet proved possible to predict unambiguously the behavior of a chiral gauge theory in the $N \to \infty$ limit [6]. We could attempt to formulate dynamical hypotheses which, together with the anomaly condition, permit us to reach conclusions about the realization of the global flavor symmetry in various chiral gauge theories [7]. But we would rather attempt to determine how flavor symmetries are realized by calculating instead of guessing. The only available calculational methods which we believe to be powerful enough to provide answers are the methods of lattice gauge theories. This, then, is the motivation for attempting to construct chiral gauge theories on the lattice.

Before going on to the formulation of chiral gauge theories on the lattice, we will briefly review an example of a chiral gauge theory [1, 2]. The example serves to clarify the dynamical issues. We shall also be referring back to it later, because much of our discussion of lattice chiral gauge theories will be carried out, for the sake of definiteness, in the context of this example.

In this example, the gauge group is $G = SU(5)$ and the fermions are left-handed, two-component spinors,

$$\varphi_{ij}\,, \qquad \psi^{i}\,, \tag{1.1}$$

transforming as the representations 10 and $\bar{5}$ respectively of SU(5). (Note that SU(5) anomalies cancel for this choice of the fermion representation content.) This is a chiral gauge theory, because no SU(5)-singlet Lorentz-invariant fermion bilinear can be constructed from φ_{ij} and ψ^{i}.

The global flavor symmetry of this model is $G_f = U(1)$, where the fermions carry U(1) charges

$$Q_\varphi = 1\,, \qquad Q_\psi = -3\,. \tag{1.2}$$

(The independent U(1) symmetry is destroyed by SU(5) instanton effects.) We would like to know how this global U(1) symmetry is realized.

One particularly simple, and therefore plausible, possibility is that the U(1) symmetry is unbroken, and a massless composite fermion couples to the SU(5)-singlet operator $B = \varphi_{ij}\psi^{i}\psi^{j}$, which carries the U(1) charge $Q_B = -5$. It is easily verified that this realization of the U(1) symmetry satisfies the 't Hooft anomaly conditions,

$$\begin{aligned} Q_B &= 10Q_\varphi + 5Q_\psi\,, \\ (Q_B)^3 &= 10(Q_\varphi)^3 + 5(Q_\psi)^3\,. \end{aligned} \tag{1.3}$$

But it is also possible that the U(1) symmetry is spontaneously broken. Spontaneous symmetry breakdown would be signaled by a vacuum expectation value for an SU(5)-singlet operator which carries a U(1) charge, such as BB or $(\varphi\psi)^5$. General algebraic arguments do not allow us to determine which of these possibilities is favored by the dynamics of this model. It is for this purpose that we wish to study the SU(5) model, and other chiral gauge theories, on the lattice.

In sect. 2, we review the technical difficulty, the fermion doubling problem, which arises when a fermion field is defined on a lattice, and in sect. 3 we recall a method developed by Wilson for dealing with the doubling problem in QCD. In sect. 4, we propose a generalization of Wilson's method which can be applied to chiral gauge theories and illustrate its use in the SU(5) model described above. We formulate a strong-coupling expansion for the SU(5) model in sect. 5. Sect. 6 contains speculations about the continuum limit of a chiral gauge theory. Sect. 7 contains conclusions. In appendix A the strong-coupling expansion is described in greater detail, and in appendix B, we explain how our methods can be applied to other examples of chiral gauge theories.

2. Lattice fermions

A technical difficulty arises when fermion fields are defined on a spatial lattice. The difficulty, the fermion "doubling" problem, may be understood in several different ways. Its essence can be appreciated if we consider the discrete Dirac equation for a free massless fermion in one spatial dimension [8].

This equation has the form

$$\dot{\psi}_n = \frac{\alpha}{2a}(\psi_{n+1} - \psi_{n-1}), \tag{2.1}$$

where n is an integer which labels the lattice sites, a is the lattice spacing and α is a Dirac matrix which has eigenvalues ± 1. The Fourier transform of this equation for an eigenstate of α with eigenvalue $+1$ is

$$\omega = -\frac{\sin ka}{a}. \tag{2.2}$$

The Fourier modes which survive in the continuum limit are those which have finite ω as a approaches zero. In addition to the "ordinary" modes with $ka \sim 0$, which have negative group velocity and therefore correspond to left-moving fermions, the surviving modes include the "mirror" modes with $ka \sim \pi$, which have positive group velocity and correspond to right-moving fermions. These mirror fermions are especially troublesome if we wish to construct theories with gauged chiral symmetries. For each left-moving fermion carrying given quantum numbers in the theory, there will appear uninvited a right-moving fermion with the same quantum numbers. A theory which was intended to be chiral becomes vector-like.

It is instructive to note that the doubling of fermion modes can be understood as a consequence of a symmetry, a discrete symmetry of the lattice action [9–11]. For example, the lattice action for a free Dirac fermion is

$$S_0 = \frac{1}{2a}\sum_{n,\hat{\mu}} \bar{\psi}_n \gamma^\mu (\psi_{n+\hat{\mu}} - \psi_{n-\hat{\mu}}) + m \sum_n \bar{\psi}_n \psi_n , \tag{2.3}$$

which is invariant under the symmetry operations,

$$\begin{aligned} T_\mu \psi_n &= (-1)^{n_\mu}, \\ \bar{\psi}_n T_\mu^{-1} &= \bar{\psi}_n \gamma_5 \gamma_\mu (-1)^{n_\mu}. \end{aligned} \tag{2.4}$$

Each of the symmetry operators T_μ, when acting on a long-wavelength mode with

$k_\mu a \to 0$, produces a mirror mode with $k_\mu a \sim \pi$. Thus the discrete symmetry requires ordinary modes and mirror modes to have equal mass and to couple to all flavor symmetry currents with equal strengths, so that mirror modes remain coupled to all long-wavelength physics, and survive in the continuum limit. In four spacetime dimensions the number of fermion species is increased sixteenfold.

In fact, this spectrum doubling is not just a property of the action eq. (2.3), but a quite general property of lattice fermion theories [12]. Perhaps the deepest and most general way of understanding the reason for the survival of the mirror modes emerges when we consider the anomalous global flavor symmetries of a given model [10, 11, 13]. For example, consider QCD with n massless quark flavors, for which the axial $SU(n)_L \times SU(n)_R$ currents can be seen by a continuum perturbation theory calculation to have anomalies [14]. The Ward identities derived from the conserved $SU(n)_L \times SU(n)_R$ currents are, however, satisfied without any anomalies in the lattice theory. Anomalies are associated with short-distance ambiguities [15], and these do not occur in the lattice theory, because it is perfectly well-defined at short distances. The continuum limit of the lattice theory therefore has no $SU(n)_L \times SU(n)_R$ anomalies, and mirror fermions must survive in the continuum limit in order to cancel the anomalies of the ordinary fermions. This argument demonstrates that fermion doubling is not a shortcoming of any particular type of discretization method, but a very general disease associated with any reasonable means of regularizing the short-distance behavior of a theory with continuous chiral symmetries.

It has been suggested that lattice regularization which preserves chiral symmetry and avoids fermion doubling can be achieved in a lattice theory with a nonlocal action [16]. However, this claim is difficult to verify in perturbation theory [17]; perturbation theory cannot be implemented, because of infrared singularities, unless the nonlocality is smoothed out, which then restores the fermion doubling. Furthermore, a nonperturbative version of the anomaly argument described above indicates that fermion doubling is not really avoidable unless the chiral symmetry is explicitly broken [18]. Evaluating whether the method of ref. [16] can really be used to construct a continuum theory with an undoubled fermion spectrum appears to involve subtleties associated with taking the large volume limit [19]. We will not consider these issues further in this paper.

3. Wilson's method

The fermion doubling problem is especially severe in a chiral gauge theory, because the unwanted mirror modes transform a theory which was intended to be chiral into a vector-like theory. But it occurs, in a milder form, even in QCD. If we naively construct lattice QCD with one fermion flavor, we obtain a theory of sixteen flavors (in four spacetime dimensions). Wilson [20] has proposed a method for eliminating the extra fermion modes in QCD, which has been adopted in some numerical calculations [21].

Wilson suggests that a perturbation be added to the QCD lattice action of the form

$$S' = a \sum_n \bar{\psi}_n \partial^2 \psi_n \,, \tag{3.1}$$

where ∂^2 represents the discrete laplacian operator

$$\partial^2 \psi_n = \frac{1}{a^2} \sum_{\hat{\mu}} \left(\psi_{n+\hat{\mu}} + \psi_{n-\hat{\mu}} - 2\psi_n \right) . \tag{3.2}$$

(We have replaced link variables by 1 in eq. (3.1).) Because (3.1) is a dimension-5 operator in the continuum limit, one might expect it to become an irrelevant operator in the infrared, and not affect the continuum limit at all. This expectation may be justified for the ordinary modes with $ka \sim 0$, but for the mirror modes with $ka \sim \pi$, this operator is not irrelevant. In fact, it generates an effective mass M for the mirror modes of order $ak^2 \sim 1/a$ which splits the mirror modes from the ordinary modes, and causes the mirror modes to decouple in the continuum limit.

The perturbation (3.1) can generate masses for the mirror modes which split these modes from the ordinary modes only because it explicitly breaks both chiral symmetry and the discrete symmetries (2.4). Once chiral symmetry is explicitly broken, nothing can prevent the dimension-three operator

$$S_m = m \sum_n \bar{\psi}_n \psi_n \tag{3.3}$$

from being induced. This operator is relevant for even the ordinary fermion modes. If we wish to obtain chiral-invariant QCD in the continuum limit we must carefully tune one free parameter in our lattice action in order to guarantee that m approaches zero as $a \to 0$. That such tuning is required to attain a chiral-invariant continuum theory should not be regarded as a serious shortcoming of Wilson's method. QCD contains a free parameter, an intrinsic quark mass, and the chiral-invariant theory corresponds to a particular choice of this parameter, namely zero. To pick out the chiral-symmetric theory we must adjust a knob to set the bare mass equal to zero. In practical calculations, one may use Wilson's method to determine whether chiral symmetry is spontaneously broken in the continuum theory. That is, one can check that it is possible by tuning one parameter to choose the pion mass to be zero, and that the massless pion obeys the low-energy theorems which should be satisfied by the Goldstone boson of chiral symmetry.

It is also possible to understand how the anomalies of chiral $SU(n)_L \times SU(n)_R$ are restored in the continuum limit of the lattice theory [10, 11, 22]. Matrix elements of the divergences of the chiral symmetry currents $J^a_{\mu L}$ and $J^a_{\mu R}$ couple to the nearly massless ordinary fermions as well as to the mirror fermions with masses $M \sim 1/a$. As $M \to \infty$, the mirror fermions decouple, but as they do so they reproduce the local anomaly terms. Indeed, one method used to compute the anomaly in continuum perturbation theory was to introduce a heavy regulator fermion to control linear ultraviolet divergences and then allow the regulator fermion mass to approach infinity [14]. In the lattice theory, mirror fermions assume the role of regulator fermions.

4. A generalization of Wilson's method

We see that Wilson's method allows us to take a continuum limit of lattice QCD which has all the desired properties of the continuum theory, including the desired number of light fermions. Now, what about theories which are not vector-like, but instead have fermions in a complex representation of the gauge group? For these chiral gauge theories, the doubling problem is especially delicate, for the presence of the mirror modes in the continuum limit alters the theory in an essential way. It converts a theory which was intended to be chiral into a vector-like theory.

Wilson's method cannot be straightforwardly extended to chiral gauge theories, because it is not possible to construct gauge-invariant, Lorentz-invariant bilinears for all the fermions in a chiral theory. A term of the form (3.1), giving masses of order $1/a$ to all of the mirror fermions, would necessarily break the gauge symmetry.

One might hope that it is possible to arrange in a chiral lattice gauge theory, without any explicit breaking of gauge invariance, for the mirror modes to obtain large masses spontaneously [23]. But this approach is unlikely to yield the desired continuum limit, for two reasons [11]. First, the large masses of the mirror fermions are bound to be fed down, through gauge boson exchange effects, to the ordinary fermions modes, and it is not clear how these masses can be tuned to zero without introducing non-gauge-invariant counterterms. Second, spontaneous generation of fermion masses means, in a chiral gauge theory, spontaneous breakdown of the gauge symmetry. Unless, as seems unlikely, it is possible to arrange by a suitable tuning of parameters for the Goldstone bosons to decouple, some of the gauge bosons will acquire masses of order $1/a$. In the continuum limit the gauge group will be a subgroup of the original gauge group, under which the fermion representation is real. This will signal, not a spontaneous breakdown of the gauge symmetry due to the effects of the strong gauge coupling, but rather a failure to construct the desired continuum theory.

We propose an extension of Wilson's method which eschews both explicit and spontaneous breaking of the gauge symmetry. We intend to add a perturbation to the lattice action of a chiral gauge theory, analogous to Wilson's term, which will cause the mirror fermion modes to decouple in the continuum limit.

As will be explained in sects. 5 and 6, this perturbation will bind composite fermions which have appropriate quantum numbers to pair up with the unwanted mirror fermions and give them large gauge-invariant masses. But, before considering the effects of the perturbation in detail, let us list some of the general properties that it must have, and see how it is constructed.

Our discussion of the fermion doubling problem in sects. 2 and 3 suggests that we should require that this perturbation satisfy the following four conditions. First it must be gauge invariant. Second, it must break explicitly all global flavor symmetries which have anomalies. Third, it must break explicitly any discrete symmetries which relate the mirror modes to the ordinary modes. Fourth, it must be "relevant" for the mirror modes, but "irrelevant" for the ordinary modes. Wilson's term (3.1) for QCD has all these properties. In a chiral gauge theory a term satisfying these four requirements necessarily involves multifermion couplings or couplings of fermion bilinears to auxiliary scalar fields which aslo couple to the gauge fields. In either case, the hypotheses of the no-go theorem of Nielsen and Ninomiya [12] are not satisfied.

So far, we have only explained why we believe that these four conditions are necessary, but we have not explained how a perturbation satisfying these conditions might effect the decoupling of the mirror modes in a chiral gauge theory. Postponing this explanation until later, let us first consider how the needed perturbation is constructed in a particular example - the SU(5) model described in sect. 1, with fermions

$$\begin{array}{cc} \varphi_{ij}\,, & \psi^{j} \\ 10 & \bar{5}\,. \end{array} \tag{4.1}$$

This model has a global U(1) flavor symmetry, and the $U(1)^3$ anomaly can be saturated by a composite fermion which couples to the operator $B=\varphi_{ij}\psi^i\psi^j$. The simplest way to construct a perturbation satisfying our criteria is to introduce a "spectator" fermion χ which is an SU(5) singlet, and the couplings

$$S_1(r_1, r_2)=\sum_n r_1\Delta(\varphi\psi\psi\chi)_n+\text{h.c.}+\sum_n r_2\Delta(\varphi\varphi\varphi\psi)_n+\text{h.c.}\,, \tag{4.2}$$

where

$$\Delta(ABCD)_n=-\frac{1}{2a}\sum_{\pm\hat{\mu}}\left(A_{n+\hat{\mu}}U^{(A)}_{n\hat{\mu}}B_nC_nD_n+A_nB_{n+\hat{\mu}}U^{(B)}_{n\hat{\mu}}C_nD_n+\cdots-4A_nB_nC_nD_n\right). \tag{4.3}$$

(Here the link variable $U^{(A)}$ transforms as A does under the gauge group G = SU(5).)

This perturbation leaves unbroken only an SU(5) × U(1) symmetry which has no anomalies. The $\varphi\varphi\varphi\psi$ term has the same structure as is generated by an SU(5) instanton in the continuum theory; it explicitly breaks the U(1) symmetry which will acquire the familiar $\mathrm{Tr}\,(F\tilde{F})$ anomaly in the continuum limit [10, 11, 22, 24]. The $\varphi\psi\psi\chi$ term respects a U(1) symmetry under which the spectator fermion χ carries a U(1) charge opposite to that of the composite operator $B=\varphi\psi\psi$. Therefore, the $U(1)^3$ anomalies cancel.

The "derivatives" appearing in (4.3) serve two functions. First, they explicitly break the discrete symmetries which would otherwise relate the ordinary fermion modes and the mirror modes. Second, they enhance the interactions of the mirror modes relative to those of the ordinary modes.

As we will see, it is thus possible, if the coefficients r_1, r_2 in eqs. (4.2) take values in a suitable range, that the mirror modes interact sufficiently strongly to form bound states, while the ordinary modes do not. It is in this sense that the perturbation can be relevant for the mirror modes, but irrelevant for the ordinary modes.

Once we introduce $S_I(r_1, r_2)$ nothing can prevent the appearance of lower-dimensional terms which are allowed by the remaining symmetries. These terms, analogous to the mass term (3.3) in QCD, are

$$S_0(\lambda_1, \lambda_2) = \lambda_1 \sum_n (\varphi_n \psi_n \psi_n \chi_n + \text{h.c.}) + \lambda_2 \sum_n (\varphi_n \varphi_n \varphi_n \psi_n + \text{h.c.}) . \tag{4.4}$$

Thus, the complete lattice action in the SU(5) example contains the perturbations (4.2) and (4.4), in addition to usual gauge-invariant kinetic terms for the fields φ and ψ and a free kinetic term for the field χ.

The perturbations of eq. (4.2) and (4.4) superficially resemble those introduced by Swift [23] in the context of the standard SU(2) × U(1) electroweak model, so we should stress the key difference between his approach and ours. In ref. [23], mirror fermions acquire mass through the spontaneous breakdown of the electroweak gauge symmetry, and gauge bosons also acquire mass in weak coupling perturbation theory. In our approach, manifest gauge invariance is retained; in particular, gauge boson masses vanish to all orders of weak coupling perturbation theory.

We should also emphasize the important difference between our method in chiral gauge theories and Wilson's method in QCD. Our perturbation (4.2) has been constructed to explicitly nullify the general arguments given in sect. 2, which require the survival of mirror excitations in the continuum limit. Wilson's term (3.1) plays the same role in QCD. However, the effect of Wilson's term on the mirror fermion modes can be studied in weak-coupling perturbation theory [25], while in a chiral gauge theory, the mirror modes are prevented from acquiring masses by the gauge symmetry to all orders in weak-coupling perturbation theory. To claim that our method removes the fermion doubling problem, then, we must argue that weak-coupling perturbation theory is misleading, because the mirror modes are strongly coupled by the perturbation (4.2). This argument is strengthened by the strong-coupling expansion formulated in the next section.

5. The strong-coupling expansion

Our goal is to construct a theory which exhibits chiral fermion content at distances much shorter than the confinement scale of the gauge interaction. At short distances, the gauge coupling g is weak, so we may consider, as an excellent approximation to the actual theory at short distances, the case in which $g = 0$, and the only nonvanishing couplings are r_1, r_2, λ_1, λ_2 defined in eqs. (4.2), (4.4). The problem of constructing the SU(5) chiral gauge theory with the fermion content $10 + \bar{5}$ becomes, in the $g = 0$ limit, the problem of constructing a lattice theory with an exact SU(5) chiral symmetry such that, as $a \to 0$, there are massless free fermions transforming as $10+\bar{5}$ under SU(5) and all other fermion modes with SU(5) quantum numbers are pushed up to infinute mass.

In the SU(5) model, the lattice action in the $g=0$ limit becomes

$$S=\kappa_{\varphi}S_{\rm kin}(\varphi)+\kappa_{\psi}S_{\rm kin}(\psi)+\kappa_{\chi}S_{\rm kin}(\chi)+S_0(\lambda_1,\lambda_2)+S_{\rm I}(r_1,r_2)\,, \tag{5.1}$$

where $S_{\rm kin}$ is the free kinetic action of the indicated Fermi field, and S_0, $S_{\rm I}$ are given by eqs. (4.2)-(4.4), with all link variables $U_{n,\hat{\mu}}$ replaced by 1. The theory defined by (5.1) can be studied analytically in expansions about two simple limits: the weak-coupling limit $\lambda_1=\lambda_2=r_1=r_2=0$, and the strong-coupling limit $\kappa_{\varphi}=\kappa_{\psi}=\kappa_{\chi}=r_1=r_2=0$.

The weak-coupling limit defines a continuum theory with a doubled fermion spectrum. It is not the continuum theory we seek.

The strong-coupling limit is a static limit in which no propagation occurs at all; it is very distant from any continuum theory. It is nonetheless an interesting limit to study, because the degeneracy between ordinary fermion modes and mirror modes is lifted in the expansion about this limit in powers of $r_{1,2}$. We expect that the theory defined by (5.1) has a nontrivial phase structure, and that this strong-coupling expansion may provide us with information about a continuum limit in which the mirror modes do not survive. The phase structure of the theory will be further discussed in the next section. Here we explain how the strong-coupling expansion is formulated. More detail can be found in appendix A.

In the strong-coupling limit, all terms in the lattice action which link neighboring sites are dropped, and all integrals over field variables factorize into a product of integrals, each performed at a single site. The existence of a sensible strong-coupling limit places a nontrivial constraint on $S_0(\lambda_1,\lambda_2)$: the "vacuum functional" of the one-site theory will vanish by Fermi statistics unless there is a term in the expansion of e^{-S_0} in which each fermionic variable appears exactly once. With S_0 as in eq. (4.4), we see that, since φ has 10 components, ψ 5 components and χ 1 component (all times a factor of 2 for spin degeneracy), the integral of e^{-S_0} is

$$\int e^{-S_0}\propto[\lambda_1^*\lambda_1(\lambda_2^*\lambda_2)^3]^{2N}\,, \tag{5.2}$$

where N is the number of lattice sites. This strong-coupling limit is analogous to the static limit of QCD, in which the vacuum wave function is proportional to the quark mass raised to a large power. We note that both terms in eq. (4.4) must be present in order that this static limit exist.

We can now perform a strong-coupling expansion in powers of $\kappa_{\varphi,\psi,\chi}r_{1,2}$ about the static limit. To all orders in this expansion, the spectrum of the theory contains only massive states, even though the $SU(5)\times U(1)$ chiral symmetry is exact, and the elementary fermions φ, ψ, χ are able to acquire Dirac masses consistent with the $SU(5)\times U(1)$ chiral symmetry by pairing up with composite fermion states.

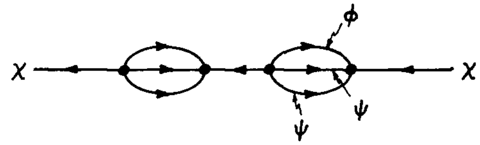

Fig. 1. Contribution to $\langle\chi_n\chi_0^\dagger\rangle$ of zeroth order in r and leading order in κ.

For example, the contribution to the propagator $\langle\chi_n\chi_0^\dagger\rangle$ to zeroth order in $r_{1,2}$ and leading order in κ is indicated in fig. 1. We see that χ propagates like a massive Dirac fermion, with the composite state $\varphi\psi\psi$ providing the required additional degrees of freedom.

To zeroth order in $r_{1,2}$ and all orders in $\kappa_{\varphi,\psi,\chi}$, the fermion mode doubling persists; all of the 16 modes associated with χ, for example, have identical propagators. The degeneracy of the ordinary modes and mirror modes is lifted in leading order in $r_{1,2}$. A contribution to $\langle\chi_n\chi_0^\dagger\rangle$ which lifts the degeneracy is shown in fig. 2. Loosely speaking, the degeneracy is lifted because modes which live on different sublattices to all orders in $\kappa_{\varphi,\psi,\chi}$ and zeroth order in $r_{1,2}$ become coupled together to first order in $r_{1,2}$. See appendix A for further details.

In perturbation theory, $r_{1,2}$ must be regarded as small, and the splittings between ordinary modes and mirror modes are small compared to the intrinsic mass scale $1/a$ of the theory. But the strong-coupling expansion demonstrates that it is possible for all the mirror modes to acquire masses consistent with the SU(5)×U(1) chiral symmetry. The key point is that the generalized "Wilson terms" we have added to the lattice action produce *bound states* which transform appropriately under SU(5)× U(1) so that Dirac masses consistent with the SU(5)×U(1) symmetry are allowed.

Our hope, now, is that it is possible by a suitable tuning of parameters to drive the masses of the ordinary modes down to zero, while maintaining splittings of order $1/a$ between the ordinary modes and the mirror modes. If so, there exists a continuum limit in which the ordinary modes survive and the mirror modes decouple. This possibility is further discussed in the next section.

A static limit and strong-coupling expansion like that presented here for the SU(5) model can be formulated for many chiral gauge theories. Other examples are discussed in appendix B.

6. The continuum limit

The outcome of our attempt to construct a chiral gauge theory as the continuum limit of a lattice theory depends on the phase structure of the lattice theory defined

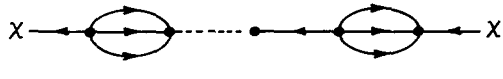

Fig. 2. Contribution to $\langle\chi_n\chi_0^\dagger\rangle$ of first order in r.

in eq. (5.1). We have not determined this phase structure. In this section, we will consider different possible types of behavior, and will discuss their implications.

This lattice theory can be studied in both weak-coupling perturbation theory and strong-coupling perturbation theory. Weak-coupling perturbation theory is an expansion in $\lambda_{1,2}$ and $r_{1,2}$, with κ taken to be of order one; to all orders in this expansion, there is a doubled massless fermion specttum. Strong-coupling perturbation theory is an expansion in κ and $r_{1,2}$, with $\lambda_{1,2}$ taken to be of order one; the fermions are massive to all orders of this expansion, and the mirror fermion modes are split from the ordinary modes in first order in $r_{1,2}$. (Note that "strong" and "weak" refer here to the value of the four-fermion coupling; the gauge coupling has been set equal to zero.)

Let us now speculate about the behavior of this theory as a function of $\lambda_{1,2}$ and $r_{1,2}$ with κ fixed at a nonzero value. In order to simplify the discussion, we will assume initially that r_1, r_2 and λ_1, λ_2 in eqs. (4.2) and (4.4) obey relations such that the action respects an SO(10) symmetry under which (χ, ψ, φ) transform as the irreducible representation 16. Hence, we may speak of a single parameter λ and a single r, and of the phase diagram in the $\lambda - r$ plane. Later, we will consider the effect of reinstating the perturbations which break the SO(10) symmetry back down to $SU(5) \times U(1)$. (We did not consider the SO(10) model from the beginning because, if we gauge SO(10), then there is no global U(1) symmetry whose realization we can investigate.)

First, we consider the phase structure along the line $r = 0$. The strong coupling expansion indicates that, for λ sufficiently large, there are massive composite fermions which are degenerate with the elementary fermions; that is, elementary two-component Weyl fermions and composite two-component Weyl fermions pair up to become massive four-component Dirac fermions. As λ decreases, one expects the composite states to become less tightly bound, and to approach threshold. At a critical coupling λ_c, the bound states disappear, and the elementary fermions become massless. For $r = 0$, the mirror fermion modes and the ordinary modes are degenerate; therefore in the continuum limit of the lattice theory with $\lambda \leq \lambda_c$ and $r = 0$, there are doubled massless fermions.

Now suppose that we allow r to assume a nonzero value. The multifermion Wilson terms give different contributions to the effective value of λ at large distances for ordinary modes and mirror modes. Therefore, we expect a wedge to open up in the $\lambda - r$ plane, as indicated in fig. 3, in between the curves along which bound states of ordinary fermions and mirror fermions approach threshold. Inside this wedge, composite states containing mirror fermion modes remain bound, so that the mirror modes are massive. But the ordinary modes are massless, and interact weakly at distance scales large compared to the lattice spacing. In the continuum limit of the lattice theory inside the wedge, all mirror fermion excitations decouple, and we obtain a theory of massless free chiral fermions! Flavor anomalies, such as the anomaly of the global U(1) current in our SU(5) example, arise in the continuum

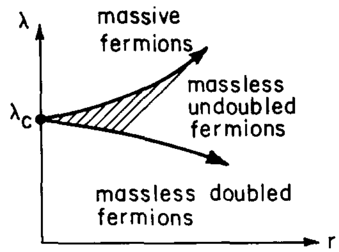

Fig. 3. Phase diagram in the $\lambda - r$ plane, assuming $\lambda_c \neq 0$. Composite fermion states go to threshold along the curves shown. In the shaded region, there is a massless undoubled fermion mode.

limit as local effects of the mirror fermions which persist as the mirror fermions decouple.

While there is good reason to believe that a nonrenormalizable coupling can produce a bound state in a cutoff theory only if the coupling strength exceeds a nonvanishing critical value [26], it is conceivable that the critical value of λ is actually $\lambda_c = 0$. In this case, the fermions have non-zero masses for arbitrarily small λ, and weak-coupling perturbation theory in λ has no radius of convergence; the approach to the $\lambda = 0$ limit is nonanalytic. If the critical value of λ is $\lambda_c = 0$ when $r = 0$, then we expect that the effective value of λ at large distances vanishes for the various fermion modes along curves in the $\lambda - r$ plane, as indicated in fig. 4. A continuum limit with undoubled massless fermions can be obtained only along these curves, rather than inside a wedge. This situation is quite reminiscent of the continuum limit in QCD with Wilson fermions.

We have described above the desired behavior of our lattice theory in the $\lambda - r$ plane, which allows the construction of a continuum theory with undoubled chiral fermions. Let us now enumerate some of the things which could go wrong; that is, circumstances which would prevent the construction of the desired chiral continuum theory.

First of all, there is a serious danger that our multifermion interactions will cause some of the exact symmetries of the lattice theory to become spontaneously broken. Indeed, there are cases in which such spontaneous symmetry breakdown is expected [26], even though there is no indication of it in either weak-coupling or strong-coupling perturbation theory [16, 27]. As we noted in sect. 4, spontaneous breakdown

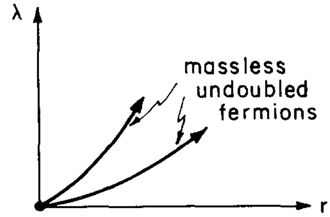

Fig. 4. Phase diagram in the $\lambda - r$ plane, assuming $\lambda_c = 0$. There are massless undoubled fermion modes along the curves shown.

of the gauge symmetry will most likely result in a failure to construct the desired continuum theory. Even spontaneous breakdown of Lorentz invariance is a possibility [28].

Even if the exact symmetries remain manifest and composite fermion states exist which become unbound at $\lambda = \lambda_c$, it might not be possible, for any choice of r, to maintain masses for the mirror fermion modes of order $1/a$ while the bound states of the ordinary fermion modes approach threshold. That this is possible seems plausible, but is not guaranteed.

Finally even if a continuum limit exists with all the properties we desire, it might be an unstable fixed point. If it is necessary to tune an infinite number of parameters to reach the desired fixed point, then our program has failed.

Is it reasonable to fear either the spontaneous breakdown of the SO(10) symmetry or the ineffectiveness of the Wilson perturbation in maintaining a splitting between ordinary modes and mirror modes near the bound state threshold? That these are genuine possibilities is better appreciated if we now consider the effect of reinstating the perturbations which explicitly break SO(10) down to SU(5)$\times$U(1); we denote by λ and r the coefficients of the quartic couplings which preserve the SO(10) symmetry, and by λ' and r' the coefficients of the corresponding couplings which break SO(10) but preserve SU(5)$\times$U(1). The point is that we can argue that our strong coupling expansion must be in some respect misleading.

In the expansion about $\lambda = \infty$, $r = \lambda' = r' = 0$, there is no indication that SO(10) becomes spontaneously broken at intermediate values of λ. Furthermore, in first order in r, the degeneracy of the ordinary and mirror modes coupling to (χ, ψ, φ) - a 16 of SO(10) - is removed. Also, in low order in λ', the degeneracy of the modes coupling to χ, ψ, and φ is removed*; this degeneracy is enforced by the SO(10) symmetry, but not by the SU(5)$\times$U(1) symmetry. But it is impossible for all these features to persist down to the critical value $\lambda = \lambda_c$ at which the bound states disappear. For if it were possible, then, by choosing $\lambda = \lambda_c$ and r, λ' small but nonvanishing, we could construct a continuum theory in which only the ordinary mode coupling to one of χ, ψ, or φ survives, and which has no SU(5)$\times$U(1) anomalies. No such continuum theory exists.

We are left to choose from among three surprising possibilities. Perhaps, for $\lambda = \lambda_c$, $r = \lambda' = r' = 0$, the SO(10) symmetry is spontaneously broken to a nonchiral symmetry. Then it is consistent for splittings of order r and order λ' to occur; the exact unbroken symmetries of the lattice theory will not become anomalous in the continuum theory. Perhaps the SO(10) symmetry (or a chiral subgroup) remains manifest at $\lambda = \lambda_c$, but there is no splitting even for nonzero r between the critical values of λ at which the bound states of ordinary modes and mirror modes disappear, contrary to our expectation based on the behavior for large λ. Or perhaps, when $\lambda = \lambda_c$, there is no order λ' splitting of the degeneracy of χ, ψ, and φ. Each of the

* We have found that the degeneracy is not removed in first order in λ', but have no reason to expect it to persist in higher orders.

latter two possibilities requires that there is a mode degeneracy which is not enforced by any apparent exact symmetry, but this requirement seems less implausible when we recall that the chiral mode doubling is already an example of such a phenomenon. Our attempt to construct a chiral gauge theory will succeed only if the third possibility is realized.

The point of the above discussion is not that it might be possible to construct a chiral gauge theory with gauge group SO(10) but not one in which the subgroup SU(5) is gauged. Clearly, If we can gauge SO(10) then we can gauge any subgroup; we need only break SO(10) to SU(5) by the Higgs mechanism. Rather, the point is that we gain insight into how our theory might behave by first considering the theory with SO(10) symmetry, and then inquiring about the effect of a perturbation which breaks SO(10) to SU(5) $\times$ U(1). It may be enlightening, in fact, to back up another step and consider a theory with an even larger symmetry.

Regarding our fields χ, ψ, φ as making up a 16 (denoted by ψ) of an exact SU(16) symmetry group, we can construct the SU(16)-invariant action

$$S(\kappa, \lambda, r) = \kappa S_{\mathrm{kin}}(\psi) + \sum_n \frac{1}{16!} \varepsilon^{a_1 \cdots a_{16}} [\lambda \psi_n^{a_1} \cdots \psi_n^{a_{16}} + r\Delta(\psi_n^{a_1} \cdots \psi_n^{a_{16}})] , \tag{6.1}$$

where

$$\Delta(\psi_n^{a_1} \cdots \psi_n^{a_{16}}) = -\frac{1}{2a} \sum_{\pm\hat{\mu}} (\psi_{n+\hat{\mu}}^{a_1} \psi_n^{a_2} \cdots \psi_n^{a_{16}} + \psi_n^{a_1} \psi_{n+\hat{\mu}}^{a_2} \cdots \psi_n^{a_{16}} + \cdots$$

$$- 16 \psi_n^{a_1} \cdots \psi_n^{a_{16}}) . \tag{6.2}$$

The strong-coupling expansion in this theory is qualitatively similar to that of the SO(10) theory; for large λ, ordinary modes and mirror modes are split in order r. Since the exact SU(16) symmetry is "anomalous" - a continuum theory with a single Weyl fermion transforming as a 16 of SU(16) has anomalies - there are three possible types of behavior of this theory that we can distinguish for $\lambda \sim \lambda_c$, the critical value of λ at which the bound states disappear.

The SU(16) symmetry might be spontaneously broken to a nonchiral symmetry (such as SO(16)) for $\lambda \sim \lambda_c$. Then we are unable to obtain a chiral continuum theory for any value of r. The SU(16) symmetry might be unbroken, or spontaneously broken to a subgroup under which the fermions have anomalies. But then we know that the Wilson term must be unable to split the ordinary and mirror modes at $\lambda \sim \lambda_c$; otherwise, we would have a lattice theory with exact SU(16) symmetry that acquired SU(16) anomalies in the continuum limit, which is impossible. So again we cannot obtain a chiral continuum theory. Finally, SU(16) might be spontaneously broken to a chiral subgroup (such as SO(10)) under which the fermions transform as a complex representation with no anomaly. Only in the third case can we succeed in constructing a chiral continuum theory. But even if the third, favorable, case is not selected in the limit of exact SU(16) symmetry we might be able to encourage the theory to realize this possibility by adding perturbations which break SU(16) but preserve, say, SO(10).

We hope, now, that the reader appreciates that there are serious questions about the correctness of the phase structure we have proposed in fig. 2, but that we are not being wildly optimistic in suggesting that a chiral continuum theory can be constructed by the method we have suggested.

So far, we have discussed our chiral theory only for the case of vanishing gauge coupling. Let us now consider how the theory is affected when the gauge coupling constant is turned on. At short distances, much less than the confinement scale of the gauge interaction, our previous picture is unmodified. The gauge coupling does not interfere with our ability to split the mirror modes from the ordinary modes, and if there is a stable fixed point with chiral fermion content when the gauge coupling vanishes, then we will still be able to reach that fixed point after the gauge coupling is turned on. As usual in an asymptotically-free lattice gauge theory, the gauge coupling will be chosen to vanish at the fixed point.

At large distances, the effective gauge coupling becomes strong, and the theory is substantially modified by the gauge interaction. Our goal, of course, has been to study this nonperturbative physics; in particular, we wish to determine the realization of the gauge symmetry and global flavor symmetries of the chiral gauge theory.

A hint concerning the realization of the global U(1) symmetry of our SU(5) model can be extracted from the expansion about infinite gauge coupling, $g=\infty$. For infinite gauge coupling, only SU(5)-singlet states can propagate. Hence, to any finite order in the hopping parameter κ, the fermion spectrum consists of modes coupling to the elementary "spectator" field χ and of modes coupling to the SU(5)-singlet composite operator $\varphi\psi\psi$. Furthermore, the U(1) global symmetry is manifest for large and small gauge coupling to any order in κ; there is no massless "spin wave" excitation. The U(1) symmetry thus has a status similar to that of the vectorlike global symmetries of QCD, treated by Wilson's method. This observation bolsters the contention that the U(1) symmetry remains manifest in the continuum limit, with the 't Hooft anomaly condition saturated by a massless fermion coupling to the composite operator $\varphi\psi\psi$.

In practice, whether we do numerical or strong-coupling calculations, we can verify that the chiral-invariant continuum limit has been reached only by checking that the particle spectrum satisfies certain requirements; hence, there is a possible ambiguity. In our SU(5) model, for example, the spectrum of the continuum theory should contain either a Goldstone boson, if the global U(1) is spontaneously broken, or a massless fermion, if U(1) is unbroken. One hopes that only one of these two possibilities can be realized by an appropriate tuning of parameters. If so, the realization of the U(1) symmetry can be determined by an analysis of the spectrum of the theory.

We do not expect this ambiguity to arise; it is merely a possibility which we cannot exclude. In the SU(5) model, the perturbation (4.4) induces a mass term in the effective lattice action coupling the spectator fermion χ to the SU(5)-singlet composite fermion operator $B=\varphi\psi\psi$. Our expectation is that this mass term can be tuned to zero, so that the spectator decouples from the chiral gauge theory. When the spectator is decoupled, B will be massless, if the global U(1) symmetry is not spontaneously broken.

Similarly, well-formulated questions about the realization of the gauge symmetry can be expressed as conditions on the spectrum, and can thus be addressed by numerical or strong-coupling lattice calculations.

7. Conclusions

There are two central questions concerning chiral gauge theories. The first question is, can continuum theories with gauged chiral symmetries be constructed at all? The second question is, how are the gauge and global symmetries of these theories realized? In this paper, we have addressed only the first question, and we have arrived at no unequivocal answer.

However, we have proposed a scheme for constructing continuum chiral gauge theories which we believe has a reasonable chance of working, and the validity of our scheme can be checked.

We have shown that it is possible, without breaking the gauge symmetry, to construct a strong-coupling limit of a lattice chiral gauge theory in which all fermions are massive, and the "mirror" fermion modes are heavier than the "ordinary" fermion modes. Elementary fermions transforming as a complex representation of the gauge group are able to acquire explicit masses consistent with the gauge symmetry by pairing up with composite fermion states transforming as the conjugate representation of the gauge group. The composite fermion states are bound, not by the gauge interaction, but by an auxiliary interaction which has been introduced for this explicit purpose.

The composite states containing ordinary fermion modes become unbound for some critical value of the coupling strength of the auxiliary interaction. We argued that, if large splittings between the ordinary modes and mirror modes can be maintaned at the critical point, then a continuum chiral gauge theory can be constructed.

In principle, Monte Carlo calculations implementing our scheme can be done, but reasonably efficient methods for dealing with fermions will be needed. One possible approach is to introduce auxiliary scalar variables so that the action can be rewritten in a form quadratic in fermionic variables, and then integrate out the fermions. The nonlocal terms in the resulting effective action are an essential part of the dynamics, and cannot be ignored. Numerical analysis of a chiral gauge theory should begin with a search for a critical point in the theory with vanishing gauge coupling.

It may also prove useful to carry out more detailed calculations in strong-coupling perturbation theory. One might gain insight into the phase structure of the ungauged theory by calculating to sufficiently high order in the expansion in κ and r. And, it may be possible to study the full gauge theory close to the critical point by doing high-order calculations, and thus obtain some information about the spectrum of the continuum theory.

Our attempts to construct chiral gauge theories on the lattice have led us to propose a complicated method involving speculative dynamics. We make this proposal because we see no alternative*. The fermion doubling problem is a generic problem for any theory which has chiral symmetry and is regulated at short distances; symmetries which are to acquire anomalies in the continuum theory must be explicitly broken in the lattice theory. Moreover, it is impossible to give mass to the mirror fermion modes in weak-coupling perturbation theory in a chiral gauge theory without explicitly breaking the gauge symmetry. Thus one is led to construct lattice theories with symmetry-breaking terms of high dimension and to seek nontrivial fixed points in these theories. In this paper, we have taken a first tentative step in this direction.

We thank Paul Ginsparg and Steve Shenker for sharing with us their insights into the fermion doubling problem.

Appendix A

THE STRONG COUPLING EXPANSION FOR THE SU(5) MODEL

In this appendix we describe in greater detail the strong-coupling expansion formulated in sect. 5. We will determine the spectrum of low-lying fermion and scalar states to leading nontrivial order in the strong-coupling expansion. Here we consider only the SU(5) model discussed in the text. Other examples are considered in appendix B.

The lattice action of eq. (5.1) for the (ungauged) SU(5) model is given more explicitly by

$$S = S_{\text{kin}}(\kappa_\varphi, \kappa_\psi, \kappa_\chi) + S_0(\lambda_1, \lambda_2) + S_1(r_1, r_2)\,, \tag{A.1}$$

where

$$S_0(\lambda_1, \lambda_2) = \lambda_1 S_{01} + \lambda_2 S_{02}$$

and

$$S_{01} = \sum_n (\varphi_{ij}^{\mathrm{T}}(n)\sigma_2\sigma_\mu \chi(n))(\psi^{i\mathrm{T}}(n)\sigma_2\sigma_\mu\psi^j(n)) + \text{h.c.}\,, \tag{A.2}$$

$$S_{02} = \sum_n \tfrac{1}{4}(\varphi_{ij}^{\mathrm{T}}(n)\sigma_2\sigma_\mu\psi^s(n))(\varphi_{kl}^{\mathrm{T}}(n)\sigma_2\sigma_\mu\varphi_{rs}(n))\varepsilon^{ijklr} + \text{h.c.}\,. \tag{A.3}$$

The kinetic terms for the fermions may be written

$$S_{\text{kin}}(\kappa_\varphi, \kappa_\psi, \kappa_\chi) = \kappa_\varphi S_{\text{kin}}(\varphi) + \kappa_\psi S_{\text{kin}}(\psi) + \kappa_\chi S_{\text{kin}}(\chi)\,,$$

where for example

$$S_{\text{kin}}(\psi) = \frac{1}{2a} \tilde{\sum_{n,\hat{\mu}}} \psi_i^\dagger(n)\sigma^\mu\psi^i(n+\hat{\mu})\,, \tag{A.4}$$

* With the possible exception of the method of ref. [16].

with $\tilde{\sum}$ defined by

$$\tilde{\sum_{\hat{\mu}}} A(n+\hat{\mu}) = \sum_{\hat{\mu}} (A(n+\hat{\mu}) - A(n-\hat{\mu}))\,, \tag{A.5}$$

where $\hat{\mu}$ is a unit shift in the positive μ direction. Finally, our multifermion "Wilson terms" are

$$S_1(r_1, r_2) = r_1 \Delta S_{01} + r_2 \Delta S_{02}\,, \qquad \text{A.6)}$$

where Δ is the shift operator introduced in eq. (4.3)

$$\begin{aligned}\Delta(A_1(n)\cdots A_l(n) = -\frac{1}{2a}\sum_{\pm\hat{\mu}} [&A_1(n+\hat{\mu})A_2(n)\cdots A_l(n)\\ &+A_1(n)A_2(n+\hat{\mu})A_3(n)\cdots A_l(n)\cdots\\ &+A_1(n)\cdots A_l(n+\hat{\mu}) - lA_1(n)\cdots A_l(n)]\,.\end{aligned} \tag{A.7}$$

It is convenient to absorb the local part of $S_1(r_1, r_2)$ into the zeroth-order action. We then obtain the new zeroth-order action

$$S_0(\Lambda_1, \Lambda_2) = \Lambda_1 S_{01} + \Lambda_2 S_{02}\,, \tag{A.8}$$

where

$$\Lambda_i = \lambda_i + \frac{16 r_i}{a} \qquad (i = 1, 2)\,. \tag{A.9}$$

We now regard the strong-coupling expansion as an expansion in κ and r with Λ held fixed

When $\kappa_\varphi = \kappa_\psi = \kappa_\chi = r_1 = r_2 = 0$ there are no couplings between neighboring sites, and the vacuum functional is simply

$$\langle e^{-S_0}\rangle \equiv \int e^{-S_0} = \prod_n \left(\int e^{-S_0(n)}\right) = \prod_n (C(\Lambda_1^*\Lambda_2)^2(\Lambda_2^*\Lambda_2)^6)\,, \tag{A.10}$$

where C is a numerical constant which results from doing the integrals over the fermionic degrees of freedom at each site. The strong-coupling limit is a static limit in which no propagation occurs. Propagation from site to site may be treated perturbatively by expanding in the "hopping parameters" κ and r.

We now turn to the analysis of the spectrum of low-lying states in the strong-coupling limit. To begin let us find the mass of the lightest states which couple to the spectator field χ. This mass can be extracted from the asymptotic large-distance behavior of the two-point function

$$S^{\alpha}_{K\beta}(n; \chi) \equiv \frac{\langle \chi^\alpha(n)\, e^{-S} \chi^\dagger_\beta(0)\rangle}{\langle e^{-S}\rangle}\,. \tag{A.11}$$

We will find the mass by solving a recursion relation for S_K which holds to leading nontrivial order in κ and r.

The recursion relation for S_K to order κ^3 and order r is

$$S^{\alpha}_{K\beta}(n)=\frac{\kappa_\varphi\kappa^2_\psi}{(2a)^3\Lambda_1}\tilde{\sum_{\hat{\mu}}}(\bar{\sigma}^\mu)^\alpha{}_{\alpha'}S^{\alpha'}_{M\beta}(n+\hat{\mu})+\frac{r_1}{2a\Lambda_1}\sum_{\pm\hat{\mu}}S^\alpha_{K\beta}(n+\hat{\mu})\,. \qquad \text{(A.12)}$$

Here S_M is defined by

$$S^\alpha_{M\beta}(n;\chi)=\langle B^{\dagger\alpha}(n)\,\mathrm{e}^{-S}\chi^\dagger_\beta(0)\rangle/\langle \mathrm{e}^{-S}\rangle\,, \qquad \text{(A.13)}$$

where B is the composite operator

$$B_\alpha(n)=\tfrac{1}{120}(\psi^{i\mathrm{T}}(n)\sigma_2\sigma_\mu\psi^j(n))(\varphi^{\mathrm{T}}_{ij}(n)\sigma_2\sigma_\mu)_\alpha\,. \qquad \text{(A.14)}$$

The origin of the two terms in eq. (A.12) is indicated in fig. 5. If the field χ sits at site n, the integration over $\chi(n)$ requires one less power of Λ_1 from the expansion of $\exp[-S_0(n)]$. The missing φ and ψ variables must hop over from neighboring sites. The contribution to S_K of lowest order in κ is generated if φ and both ψ's hop over together from the same site.

It is already suggested by eq. (A.12) that the states coupling to χ and $B^\dagger$ are mixed in lowest nontrivial order in the expansion in κ, producing a massive four-component fermion. To find the mass to lowest order in κ, we note that S_M obeys an exact recursion relation

$$S^\alpha_{M\beta}(n)=\frac{1}{120\Lambda_1}\,\delta^\alpha{}_\beta\delta_{n,0}+\frac{\kappa_\chi}{2a}\,\frac{1}{120\Lambda_1}\tilde{\sum_{\hat{\mu}}}\sigma^{\mu\alpha}{}_{\alpha'}S^{\alpha'}_{K\beta}(n+\hat{\mu})+\frac{r_1}{2\Lambda_1 a}\sum_{\pm\hat{\mu}}S^\alpha_{M\beta}(n+\hat{\mu})\,. \qquad \text{(A.15)}$$

Now, rewriting (A.12) and (A.15) in momentum space, we obtain

$$\left(\Lambda_1-\frac{r_1}{2a}\sum_{\pm\hat{\mu}}\mathrm{e}^{ip^\mu a}\right)S^\alpha_{K\beta}(p)=\frac{\kappa_\varphi\kappa^2_\psi}{(2a)^3}\left(\tilde{\sum_{\hat{\mu}}}\bar{\sigma}^{\mu\alpha}{}_{\beta'}\,\mathrm{e}^{ip^\mu a}\right)S^{\beta'}_{M\beta}(p)\,, \qquad \text{(A.16)}$$

$$\left(\Lambda_1-\frac{r_1}{2a}\sum_{\pm\hat{\mu}}\mathrm{e}^{ip^\mu a}\right)S^\alpha_{M\beta}(p)=\frac{1}{120}\left\{\delta^\alpha{}_\beta+\frac{\kappa_\chi}{2a}\tilde{\sum_{\hat{\mu}}}(\sigma^\mu)^\alpha{}_{\beta'}S^{\beta'}_{K\beta}(p)\right\}. \qquad \text{(A.17)}$$

Applying (A.16) twice and invoking (A.17) yields

$$\left(\Lambda_1-\frac{r_1}{2a}\sum_{\pm\hat{\mu}}\mathrm{e}^{ip^\mu a}\right)^2S^\alpha_{K\beta}(p)=\frac{\kappa_\varphi\kappa^2_\psi}{(1a)^3}\,\frac{1}{120}\tilde{\sum_{\hat{\mu}}}(\bar{\sigma}^\mu)^\alpha{}_{\alpha'}\,\mathrm{e}^{ip^\mu a})$$
$$\times\left[\delta^{\alpha'}{}_\beta+\frac{\kappa_\chi}{2a}\left(\tilde{\sum_{\hat{\nu}}}(\sigma^\nu)^{\alpha'}{}_{\beta'}\,\mathrm{e}^{ip^\nu a}\right)S^{\beta'}_{K\beta}(p)\right]. \qquad \text{(A.18)}$$

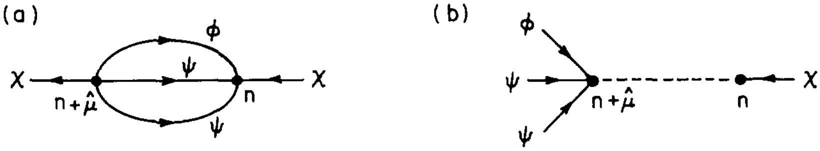

Fig. 5. Contributions to the recursion relation for $S_K(n,\chi)$ (a) of order κ^3, (b) of order r.

Inverting (A.18), we find

$$S^{\alpha}_{K\beta}(p;\chi)=\frac{\frac{i}{\kappa_\chi a}\sum_{\hat{\mu}}(\bar{\sigma}^{\mu})^{\alpha}{}_{\beta}\sin p^{\mu}a}{\frac{1}{a^2}\sum_{\hat{\mu}}\sin^2 p^{\mu}a+\frac{(2a)^4}{\kappa_\chi\kappa_\varphi\kappa_\psi^2}\frac{30}{a^2}\left(\Lambda_1-\frac{r_1}{a}\sum_{\hat{\mu}}\cos p^{\mu}a\right)^2}. \tag{A.19}$$

Finally we see that the field χ couples to 16 massive modes, with masses given by the locations of the poles of $S_K(p;\chi)$, or

$$m_\chi^2\simeq\frac{1}{a^2}\left[\frac{30\Lambda_1^2(2a)^4}{\kappa_\chi\kappa_\varphi\kappa_\psi^2}\left(1-\frac{r_1}{a\Lambda_1}\sum_{\hat{\mu}}\cos p^{\mu}a\right)^2\right]. \tag{A.20}$$

where each $p^{\mu}a$ is either 0 or π.

In eq. (A.20) we find justification for the comments in sect. 5. When $r_1=0$, there are 16 degenerate fermion modes, as we expected. For $r_1\neq 0$, the degeneracy is removed, and there is a unique mode of lowest mass (the "ordinary" mode) which is split from the nearest mirror modes by an amount proportional to $2r_1/\Lambda_1 a$. To this order of perturbation theory, we may tune r_1 so that the ordinary χ mode becomes massless while all mirror modes remain massive.

Proceeding in exactly the same way, we may now find the spectrum of low-lying states which couple to the fields ψ and φ. The only new complication is that in each case the field appears in both terms of the unperturbed action S_0 and of S_I. Therefore, the states which couple to ψ (or φ) are a linear combination of two different composite operators. For ψ the two composite operators $B_1^\dagger$ and $B_2^\dagger$ are given by

$$B_{1i\alpha}(n)=(\varphi^{\mathrm{T}}_{ji}(n)\sigma_2\sigma_\mu\chi(n))(\psi^{\mathrm{T}j}(n)\sigma_2\sigma_\mu)_\alpha\,,$$
$$B_{2i\alpha}(n)=\tfrac{1}{4}(\varphi^{\mathrm{T}}_{jk}(n)\sigma_2\sigma_\mu\varphi_{li}(n))(\varphi^{\mathrm{T}}_{rs}(n)\sigma_2\sigma_\mu)_\alpha\varepsilon^{jklrs}\,. \tag{A.21}$$

To simplify our remaining analysis we will choose to consider the special case where $\kappa_\psi=\kappa_\varphi=\kappa_\chi\equiv\kappa$, $\Lambda_1=\Lambda_2\equiv\Lambda$, and $r_1=r_2\equiv r$ in eq. (A.1). Then the fields ψ^i, φ_{ij}, and χ can be combined to form a single 16-dimensional spinor representation of O(10) and the action is invariant under O(10) symmetry. In particular the two SU(5)×U(1) invariants S_{01} and S_{02} given in eqs. (A.2) and (A.3) combine to form a single quartic invariant of O(10). In this case, the linear combination of $B_1^\dagger$ and $B_2^\dagger$ which mixes with the ψ field is just

$$B_1^{\dagger i\alpha}(n)+B_2^{\dagger i\alpha}(n)\,.$$

It is clear that for the O(10) invariant action, the previous analysis for the χ field applies. We can immediately conclude that each of the ψ^i and φ_{ij} fields couples to an associated composite operator to give 16 massive modes with masses $m_\psi^2 = m_\varphi^2 = m_\chi^2$, where

$$m_\chi^2 = \frac{1}{a^2}\left[30\Lambda^2\left(\frac{2a}{\kappa}\right)^4\left(1-\frac{r}{2\Lambda a}\sum_{\hat\mu}\cos p^\mu a\right)^2\right], \tag{A.22}$$

to lowest order in κ and r, as follows from eq. (A.20). For $r \neq 0$ there is a unique mode of lowest mass (the "ordinary" mode) which is split from the nearest mirror modes by an amount proportional to $2r/\Lambda a$.

In addition to the elementary and composite fermion states, there are composite scalar states in the strong-coupling limit. Returning to the SU(5) invariant action of eq. (A.1); we consider, for example, the composite scalar state with quantum numbers of $(\chi^{\mathrm{T}}\sigma_2\psi^i)$. We may use the operator

$$A^i(n) \equiv c_1\chi^{\mathrm{T}}(n)\sigma_2\psi^i(n) + c_2\psi_k^\dagger(n)\sigma_2\varphi^{\dagger\mathrm{T}ik}(n) \tag{A.23}$$

as an interpolating field for this scalar state. To find the mass we consider the Green function

$$G(n) = \langle A^i(n)\, \mathrm{e}^{-S}A_j^\dagger(0)\rangle / \langle \mathrm{e}^{-S}\rangle\,. \tag{A.24}$$

To lowest non-trivial order in κ, this Green function satisfies the recursion relation

$$G(n) = \frac{2}{5}\,\frac{\kappa_\psi^2\kappa_\chi\kappa_\varphi}{\Lambda_1}\,\delta_{n,0}\delta^i{}_j + \frac{1}{5}\sqrt{\frac{\kappa_\psi^2\kappa_\chi\kappa_\varphi}{2(2a)^4}}\,\frac{1}{\Lambda_1}\sum_{\pm\hat\mu} G(n+\hat\mu)\,, \tag{A.25}$$

where $c_1 = \sqrt{\kappa_\psi\kappa_\chi}$ and $c_2 = \sqrt{\tfrac{1}{2}\kappa_\varphi\kappa_\psi}$. Solving for the momentum space propagator $G(p)$, we find

$$G(p) = \frac{32\sqrt{\tfrac{1}{2}\kappa_\psi^2\kappa_\varphi\kappa_\chi}\;\delta^i{}_j}{\dfrac{4}{\alpha}\sum_\mu \sin^2(\tfrac{1}{2}p_\mu a) + \dfrac{8}{a^2}\left[5\Lambda_1\sqrt{\dfrac{2a^4}{\kappa_\psi^2\kappa_\chi\kappa_\varphi}} - 1\right]}\,. \tag{A.26}$$

Thus we see that there is a composite scalar state in the $\bar{5}$ representation of SU(5) with mass

$$m^2 \simeq \frac{8}{a^2}\left[5\Lambda_1\sqrt{\frac{2a^4}{\kappa_\psi^2\kappa_\varphi\kappa_\chi}} - 1\right], \tag{A.27}$$

to leading order in κ. The leading contribution to the recursion relation involving r is of order $\kappa^2 r$, and we are therefore justified in neglecting it. The contributions of this order have no special status for the scalar states, because there is no degeneracy to be removed by S_{I}.

By the same reasoning, we find that there are massive composite scalars which couple to each of the "spin-zero" fermion bilinears. In the special case of O(10) symmetry these scalars transform as the 10 and 126 representations.

In strong coupling perturbation theory, Yukawa couplings will also be induced between the fermion and scalar states described above. In fact, a strong coupling limit similar to the one discussed here could have been constructed if we had introduced into the lattice action, instead of multifermion interactions, Yukawa couplings of the fermions to elementary scalars in appropriate representations of SU(5). Then S_{I} would be replaced by terms of the form

$$-\frac{r_1}{2a}\sum_{\hat{\mu}} A_i(m)[\chi^{\mathrm{T}}(n+\hat{\mu})\sigma_2\psi^i(n)+\chi^{\mathrm{T}}(n)\sigma_2\psi^i(n+\hat{\mu})-2\chi^{\mathrm{T}}(n)\sigma_2\psi^i(n)]+\text{h.c.}\,,$$

where $A_i(n)$ denotes the scalar field transforming as the 5 representation of SU(5).

We conclude with a comment about the effect of higher-order corrections to the strong-coupling limit. The degeneracy of the fermion modes, broken in order r, will not be restored in any finite order of perturbation theory. But higher order corrections will generate structure for the composite states. We expect the lightest, most weakly bound, composite state, that whose constituents are "ordinary" fermion modes, to have the largest characteristic size, a size roughly inversely proportional to its mass. Therefore, as that bound state approaches threshold, it effectively disappears from the spectrum, if we confine our attention to physics at some fixed finite distance scale.

Appendix B

OTHER CHIRAL GAUGE THEORIES

In this appendix, we discuss briefly some other examples of chiral gauge theories. For each example, we construct the additional terms of the lattice action which are needed to remove the degeneracy of the ordinary fermion modes and the mirror modes. The static limit of each example and the strong-coupling expansion about that limit are described. We also consider an example of a lattice action for which no static limit exists.

Example 1. Our first example is a simple generalization of the SU(5) model [2]. The gauge group is SU(N) and the fermions transform as

$$\underset{\varphi_{ij}}{\begin{smallmatrix}\square\\\square\end{smallmatrix}}+(N-4)\underset{\psi^i_\alpha}{\bar{\square}}\,, \tag{B.1}$$

a representation free of SU(N) anomalies. (Here ij are SU(N) indices and $a=1,\cdots,N-4$ is a flavor index.) This model respects a global flavor symmetry group

$$\mathrm{G}_f=\mathrm{SU}(N-4)\times\mathrm{U}(1)\,, \tag{B.2}$$

where the U(1) charge assignments are

$$Q_\varphi=N-4\,,\qquad Q_\psi=-(N-2) \tag{B.3}$$

and the anomalies of the global symmetry currents are given by

$$\mathrm{Tr}\, Q = -\tfrac{1}{2}N(N-3)(N-4)\,, \qquad \mathrm{Tr}\, Q^3 = -\tfrac{1}{2}N^3(N-3)(N-4)\,,$$
$$\mathrm{Tr}\,(QQ_aQ_b) = -N(N-2)\delta_{ab}\,, \qquad \mathrm{Tr}\, Q_aQ_bQ_c = Nd_{abc}\,, \tag{B.4}$$

where $Q_{a,b,c}$ are SU$(N-4)$ generators.

We wish to construct a lattice action for this model which explicitly breaks all global symmetries which have anomalies. The construction proceeds as in the SU(5) model. First, we observe [2] that there is an SU(N)-singlet composite operator which satisfies the anomaly matching conditions; that is, which reproduces the flavor anomalies of eq. (B.4). It is

$$B^{\alpha}_{ab}(n) = (\varphi^{\mathrm{T}}_{ij}(n)\sigma_2\sigma^{\mu})^{\alpha}(\psi^{\mathrm{T}i}_{a}(n)\sigma_2\sigma_{\mu}\psi^{j}_{b}(n)), \tag{B.5}$$

which transforms as $(\square\square, -N)$ under $\mathrm{G}_f = \mathrm{SU}(N-4)\times \mathrm{U}(1)$.

Next, we introduce a spectator fermion χ which is a singlet under the gauge group $\mathrm{G} = \mathrm{SU}(N)$, and transforms under G_f as $(\overline{\square\square}, N)$, the representation conjugate to that according to which the composite operator B transforms. Then the interaction

$$S_{01}(n) = \chi^{ab}_{\alpha}(n)B^{\alpha}_{ab}(n) + \mathrm{h.c.}\,, \tag{B.6}$$

respects an SU$(N-4)\times$U(1) symmetry group which has no anomalies.

Finally, we must explicitly break the "axial" U(1) global symmetry which is to be spoiled by instanton effects in the continuum theory. For this purpose, we introduce an interaction

$$S_{02}(n) = [\varphi(n)^{N-2}\psi_{a_1}(n) - \psi_{a_{N-4}}(n)]\varepsilon^{a_1\cdots a_{N-4}} + \mathrm{h.c.} \tag{B.7}$$

(Color and spin indices are suppressed in (B.7).) The lattice action for this (ungauged) SU(N) model, generalizing eq. (A.1), becomes

$$S = \lambda S_0 + \kappa S_{\mathrm{kin}} + rS_{\mathrm{I}}\,, \tag{B.8}$$

where

$$\lambda S_0 = \sum_n [\lambda_1 S_{01}(n) + \lambda_2 S_{02}(n)] \tag{B.9}$$

and

$$rS_{\mathrm{I}} = \sum_n [r_1 \Delta S_{01}(n) + r_2 \Delta S_{02}(n)]\,, \tag{B.10}$$

with Δ defined as in eq. (A.7).

Let us now consider the static limit ($\kappa = r = 0$) of this lattice theory. The condition for a nontrivial vacuum functional in the static limit is

$$\langle \mathrm{e}^{-\lambda S_0}\rangle = \prod_n \left(\int_n \mathrm{e}^{-\lambda S_0(n)}\right) \neq 0\,. \tag{B.11}$$

For this condition to be satisfied, there must be a term in the expansion of $e^{-\lambda S_0}$ in which every fermionic variable appears exactly one. That is, there must exist integers n_1 and n_2 which satisfy

$$\begin{aligned} \dim \chi &= \tfrac{1}{2}(N-3)(N-4) = n_1\,, \\ \dim \varphi &= \tfrac{1}{2}N(N-1) = n_1 + (N-2)n_2\,, \\ \dim \psi &= N(N-4) = 2n_1 + (N-4)n_2\,. \end{aligned} \tag{B.12}$$

Eq. (B.12) is solved by $n_1 = \frac{1}{2}(N-3)(N-4)$, $n_2 = 3$, and a detailed check confirms that, indeed,

$$0 \neq \int_n e^{-\lambda S_0(n)} \propto (\lambda_1^* \lambda_1)^{2n_1} (\lambda_2^* \lambda_2)^{2n_2}\,. \tag{B.13}$$

Thus, a nontrivial static limit exists.

A strong coupling expansion in κ and r can now be formulated, as for the SU(5) model. The degeneracy of the ordinary fermion modes and the mirror excitations will be lifted in order r.

Example 2. The gauge group of this model is $G = SU(3)$, and the fermions transform as the anomaly-free representation

$$\underset{\varphi^i_{jk}}{\yng(3,1)} + 2\,\underset{\psi^{lm}_a}{\overline{\yng(2)}}\,. \tag{B.14}$$

This model is the simplest example of a chiral gauge theory, that is, it is the chiral gauge theory with the smallest number of degrees of freedom. We therefore regard it as a leading candidate for analysis by Monte Carlo techniques.

The global flavor symmetry group is

$$G_f = SU(2) \times U(1) \tag{B.15}$$

where the U(1) charge assignments are

$$Q_\varphi = 1\,, \qquad Q_\psi = -2\,, \tag{B.16}$$

and the anomalies of the flavor currents are

$$\mathrm{Tr}\, Q = -9\,, \qquad \mathrm{Tr}\, Q^3 = -81\,, \qquad \mathrm{Tr}\, Q Q_a Q_b = -12\delta_{ab}\,. \tag{B.17}$$

Again, we find a G-singlet composite operator which satisfies the G_f anomaly conditions, namely

$$B^\alpha_{ab}(n) = (\varphi^{\mathrm{T}i}_{jk}(n)\sigma_2\sigma^\mu)^\alpha (\psi^{\mathrm{T}jr}_a(n)\sigma_2\sigma_\mu\psi^{ks}_b(n)\varepsilon_{irs})\,, \tag{B.18}$$

which transforms as ($\yng(2)$, -3) under G_f. If we introduce the SU(3)-singlet spectator fermion χ coupled to B by

$$S_{01}(n) = \chi^{\mathrm{T}ab}_\alpha B^\alpha_{ab} + \text{h.c.}\,, \tag{B.19}$$

then S_{01} respects an $SU(2) \times U(1)$ symmetry group which has no anomalies.

Now we must explicitly break the "axial" U(1) symmetry. An SU(3) instanton would generate the operator $\varphi^{20}\psi^{10}$, but this operator can be reduced to a product of $SU(3)\times G_f$ singlets of the form $[(\varphi\varphi\psi)^2]^5$. Hence we can break the unwanted U(1) symmetry with the $SU(3)\times G_f$ invariant interaction

$$S_{02}(n) = \varepsilon^{ab}(\varphi^{i}_{jk}(n)\varphi^{i'}_{j'k'}(n)\psi^{kk'}_{a}(n))(\varphi^{j}_{ir}(n)\varphi^{j'}_{i'r'}(n)\psi^{rr'}_{b}(n)) + \text{h.c.} \qquad \text{(B.20)}$$

(Spin indices are suppressed in (B.20).) This interaction breaks a Z_5 subgroup of the "axial" U(1) which would survive in the presence of instantons.

Our lattice action for this model is

$$S = \lambda S_0 + \kappa S_{\text{kin}} + rS_{\text{I}}\,, \qquad \text{(B.21)}$$

where

$$\lambda S_0 = \sum_n [\lambda_1 S_{01}(n) + \lambda_2 S_{02}(n)]\,, \qquad \text{(B.22)}$$

$$rS_{\text{I}} = \sum_n [r_1 \Delta S_{01}(n) + r_2 \Delta S_{02}(n)]\,. \qquad \text{(B.23)}$$

The condition for the existence of a nontrivial static limit ($\kappa = r = 0$) of this model, $0 \neq \int_n \mathrm{e}^{-\lambda S_0(n)} \propto (\lambda_1^*\lambda_1)^{2n_1}(\lambda_2^*\lambda_2)^{2n_2}$, is

$$\begin{aligned} \dim\chi &= 3 = n_1\,, \\ \dim\varphi &= 15 = n_1 + 4n_2\,, \\ \dim\psi &= 12 = 2n_1 + 2n_2\,, \end{aligned} \qquad \text{(B.24)}$$

which has the solution $n_1 = n_2 = 3$. We note that it is necessary to break the Z_5 discrete symmetry in order to ensure that a nontrivial static limit exists.

Example 3. In this example, the gauge group is $G = SU(8)$ and the fermions are in an anomaly-free representation

$$\underset{\varphi_{ijk}}{\begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline\end{array}} + 5\,\underset{\psi^{i}_{a}}{\overline{\square}}\,, \qquad \text{(B.25)}$$

and the global flavor symmetry group is

$$G_f = SU(5)\times U(1)\,, \qquad \text{(B.26)}$$

where the U(1) charge assignments are

$$Q_\varphi = 1\,, \qquad Q_\psi = -3\,. \qquad \text{(B.27)}$$

It is impossible in this case to construct SU(8)-singlet composite fermion operators with quantum numbers which saturate the G_f anomaly condition. In fact, there are no SU(8)-singlet composite fermion operators at all; any operator constructed from an odd number of elementary fermion fields has odd 8-ality.

In order to construct a lattice theory in which all global flavor symmetries with anomalies are explicitly broken, we introduce the interactions

$$S_{01}(n) = (\varphi^{\mathrm{T}}_{ijk}(n)\sigma_2\varphi_{rst}(n))(\psi^{\dagger}_{m}(n)\sigma_2(\psi^{\dagger}_{l}(n))^{\mathrm{T}})\varepsilon^{ijkmlrst} + \text{h.c.}\,, \qquad \text{(B.28)}$$

$$S_{02}(n) = (\varphi^{\mathrm{T}}_{ijk}(n)\sigma_2\varphi_{rst}(n))(\varphi^{\mathrm{T}}_{mln}(n)\sigma_2\psi^{n}(n))\varepsilon^{ijkmlrst} + \text{h.c.}\,. \qquad \text{(B.29)}$$

These interactions break G_f to the anomaly-free subgroup $\mathrm{H}_f = \mathrm{O}(4)$, and also explicitly break the "axial" U(1) symmetry which is spoiled by instantons in the continuum theory.

Our lattice action for this model is

$$S = \lambda S_0 + \kappa S_{\text{kin}} + r S_{\mathrm{I}}\,, \qquad \text{(B.30)}$$

with the terms again defined as in (B.9) and (B.10). The necessary condition for the existence of a static limit is

$$\dim \varphi = 56 = 2n_1 + 3n_2\,,$$

$$\dim \chi = 40 = 2n_1 + n_2\,,$$

which is satisfied by $n_1 = 16$, $n_2 = 8$.

Example 4. Now consider a model with gauge group $\mathrm{G} = \mathrm{SU}(7)$ and fermions in the anomaly-free representation

$$\underset{\eta_{ijk}}{\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline \ \\ \hline\end{array}} + \underset{\varphi^{lm}}{\overline{\begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array}}} + \underset{\psi_r}{\square}\,. \qquad \text{(B.31)}$$

The global flavor symmetry group of this model is

$$\mathrm{G}_f = \mathrm{U}(1)_1 \times \mathrm{U}(1)_2\,, \qquad \text{(B.32)}$$

where the charge assignments are

$$(Q_1, Q_2)_\eta = (1, 4)\,,$$

$$(Q_1, Q_2)_\varphi = (-3, -7)\,,$$

$$(Q_1, Q_2)_\psi = (5, -5)\,. \qquad \text{(B.33)}$$

To construct a lattice theory in which all global flavor symmetries with anomalies are explicitly broken, we first note that the $\mathrm{U}(1)_1$ anomaly-matching condition is satisfied by the composite operator

$$B^{\alpha}(n) = (\varphi^{\mathrm{T}ij}(n)\sigma_2\sigma^{\mu})^{\alpha}(\psi^{\mathrm{T}}_{i}(n)\sigma_2\sigma^{\mu}\psi_j(n))\,,$$

which has $\mathrm{U}(1)_1$ charge

$$(Q_1)_B = 7\,.$$

We therefore introduce an SU(7)-singlet spectator fermion χ coupled to B by

$$S_{01}(n) = \chi^{\mathrm{T}}_{\alpha}(n)B^{\alpha}(n) + \text{h.c.}$$

The $U(1)_2$ and "axial" U(1) symmetries may be broken by the $U(1)_1$-invariant interactions

$$S_{02}(n) = (\eta^{\mathrm{T}}_{ijk}\sigma_2\psi_r(n))(\varphi^{\mathrm{T}ij}(n)\sigma_2\varphi^{kr}(n)) + \text{h.c.}\,,$$

$$S_{03}(n) = \varepsilon_{ijklrst}(\eta^{\mathrm{T}}_{ijk}(n)\sigma_2\sigma^{\mu}\eta_{lrs}(n))(\eta^{\mathrm{T}}_{tuv}\sigma_2\sigma_{\mu}\varphi^{uv}) + \text{h.c.}$$

If we choose the zeroth-order action to be

$$\lambda S_0 = \sum_n [\lambda_1 S_{01}(n) + \lambda_2 S_{02}(n) + \lambda_3 S_{03}(n)]\,,$$

then the necessary condition for the existence of a static limit is

$$\dim \chi = 1 = n_1\,,$$

$$\dim \eta = 35 = n_2 + 3n_3\,,$$

$$\dim \varphi = 21 = n_1 + 2n_2 + n_3\,,$$

$$\dim \psi = 7 = 2n_1 + n_2\,,$$

which is solved by $n_1 = 1$, $n_2 = 5$, $n_3 = 10$. The strong coupling expansion about the static limit may be formulated as before.

Remarks. As the above examples make clear, the problem of constructing a lattice action which meets our criteria is closely related to the problem of finding a realization of the global flavor symmetry consistent with 't Hooft's anomaly-matching condition. We require the lattice action to be gauge-invariant and to respect no global flavor symmetries which would have anomalies in the continuum theory. To meet this requirement, we seek gauge-singlet composite fermion operators which satisfy the anomaly-matching conditions for some subgroup of the flavor group, and introduce spectator fermion fields coupled to these composite operators. We then explicitly break any remaining anomalous flavor symmetries with gauge-invariant multi-fermion "condensates."

A similar procedure would be carried out to find candidate realizations of the global flavor symmetry which are consistent with the anomaly-matching condition in the continuum theory. In fact, we have performed such an analysis for a number of chiral gauge theories, as reported in ref. [7]. The formulations of the lattice theories described in examples 1-4 are a by-product of that analysis.

In each of examples 1-4, we discovered that the simplest lattice theory in which all anomalous symmetries (including the axial U(1)) were explicitly broken had a static limit. Since the existence of a static limit is a nontrivial algebraic constraint, this discovery is a bit of a surprise. It applies to many other, but not all, of the models we have analyzed. In some cases we have found it convenient to introduce interactions in which spectator fermion fields appear nonlinearly in order to ensure the existence of a static limit.

Example 5. To explain what is meant by the nonexistence of a static limit in a lattice theory, we describe here a simple toy model which suffers from this disease.

Consider a theory defined on a one-dimensional lattice, with action

$$S = \sum_n [S_{\text{kin}}(n) + S_0(n)] ,$$

where

$$S_{\text{kin}}(n) = \frac{\kappa}{2a} \sum_{i=1}^{3} a_i^\dagger(n)[a_i(n+1) - a_i(n-1)] ,$$

$$S_0(n) = a_1(n)a_2(n) + a_2(n)a_3(n) + a_3(n)a_1(n) + \text{h.c.}$$

In the static limit, $\kappa = 0$, the vacuum-to-vacuum amplitude clearly vanishes,

$$\langle e^{-S_0} \rangle = \prod_n \int \prod_{i=1}^{3} da_i(n)\, da_i^\dagger(n)\, e^{-S_0(n)} = 0$$

because each term in the expansion of $e^{-S_0(n)}$ contains an even number of $a_i(n)$'s and $a_i^\dagger(n)$'s.

Now consider the quantum mechanical system with hamiltonian $H = -S_0(n)$. This hamiltonian acts on a Hilbert space consisting of the eight states $|m_1, m_2, m_3\rangle$ where $m_i = 0$ or 1. The eigenstates of H are easily found to be

Eigenstate	Eigenvalue
$\sqrt{\frac{1}{6}}[\pm\sqrt{3}\,|000\rangle + |110\rangle + |011\rangle + |101\rangle]$	$\mp\sqrt{3}$
$\sqrt{\frac{1}{6}}[\pm\sqrt{3}\,|111\rangle + |001\rangle + |100\rangle + |010\rangle]$	$\mp\sqrt{3}$
$\sqrt{\frac{1}{2}}[|110\rangle - |011\rangle]$	0
$\sqrt{\frac{1}{2}}[|001\rangle - |100\rangle]$	0
$\sqrt{\frac{1}{6}}[|110\rangle + |011\rangle - 2|101\rangle]$	0
$\sqrt{\frac{1}{6}}[|001\rangle + |100\rangle - 2|010\rangle]$	0

We see that all eigenvalues of H are degenerate. This degeneracy is a consequence of a "charged conjugation" symmetry which interchanges $m_i = 0$ and $m_i = 1$, and commutes with H. In fact, the Hilbert space divides into two sectors, with even and odd fermion number, which are preserved by H, but are interchanged by "charge conjugation."

This behavior is characteristic of theories which, in our language, have no static limit. If the Hilbert space on which the static hamiltonian acts decomposes into physically indistinguishable sectors which are preserved by the hamiltonian, then our procedure for formulating an expansion about the static limit fails.

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